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On the E-optimality of some group divisible block designs

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SUMMARY

In this paper, we find E-optimal designs based on balanced block (BB) designs and some partially balanced incomplete block designs with group divisible associate scheme (GD PBIB) listed in Clatworthy (1973). As a result, we obtain partially balanced block designs with group divisible associate scheme (GD PBB) having equal or unequal treatment replications and equal or unequal block sizes (the classical GD PBIB designs have only equal treatment replications and equal block sizes).

Key words: balanced block (BB) designs, balanced incomplete block (BIB) designs, E-optimality, partially balanced block designs with group divisible association scheme (GD PBB), partially balanced incomplete block designs with group divisible association scheme (GD PBIB)

1. Preliminaries

In some biological and industrial experiments, there are situations involving block designs with a small number of block sizes and treatment replications and/or there are unequal treatment replications and block sizes. In practice, we have to use optimal block designs, among which the E-optimal designs deserve particular attention.

In the theory of experimental designs, E-optimality is often considered. Older papers about E-optimality refer to classic block designs with a constant linear model. More recent works take into consideration more complicated models and furthermore they extend the designs, e.g. they deal with row-column designs. For example E-optimal designs for linear, quadratic and cubic growth models with autocorrelated errors and with four or fewer time points from interval [0, 2] are given in Moerbeek (2005). E-optimal designs for quadratic and cubic growth curve models with correlated errors and with three and four time points on the time the interval [0, 2] are considered by Filipiak and Szczepańska (2005). Experiments with the same number of treatments, blocks and plots are considered by Filipiak and Różański (2005), where it is assumed that the response to a treatment is affected by other treatments, so the model of the experiment is an interference model with neighbor effects. The aim of this paper is to identify the structure of the left neighbor matrix of E-optimal design and to give a construction method for such a design.

Bagchi (1996) showed that nested row-column designs having 2×4 arrays as blocks and with treatments satisfying a rectangular association scheme with two rows and an odd number of columns are E-optimal designs. Bagchi (2004) also presented a general construction of group divisible (GD) designs and rectangular designs by utilizing resolvable and "almost resolvable" balanced incomplete block designs. As a special case, Bagchi (2004) obtained two classes of E-optimal block designs: GD designs with $\lambda_2 = \lambda_1 + 1$ and rectangular designs with two rows and $\lambda_3 = \lambda_2 - 1 = \lambda_1 + 1$, where two treatments occur together in exactly λ_i blocks (i =1, 2 for GD design and i = 1, 2, 3 for rectangular design). The numbers λ_i are called the coincidence numbers of the design. In the first class Bagchi (2004) constructed a few new E-optimal GD designs with replication number $r \geq 12$ and therefore in this paper we do not use the above designs (in Clathworthy (1973), E-optimal GD designs have $r \leq 10$).

Wallis (1996) showed equivalence between graphs and E-optimality. He gave criteria for existing E-optimal block designs. Morgan (2007) showed equivalence between designs and graphs finding 89 E-optimal block designs for up to 15 treatments. The problem of constructing E-optimal designs from an irregular BIB designs setting is studied by Morgan and Reck (2007). They found an E-optimal design for 15 treatments in 21 blocks of size 5. E-optimality for three treatments in an n-way heterogeneity setting is studied by Parvu and Morgan (2007). Brzeskwiniewicz (1989, 1995) presented a certain E-optimality criterion for block designs and two-way elimination of heterogeneity designs, respectively.

In the statistical literature investigating E-optimality of block designs, the most frequently analysed designs are those with equal replications of treatments and equal block sizes. Bagchi (2004) discussed GD designs and rectangular designs. In the present paper, we construct E-optimal designs, where the treatment replications do not have to be equal and the designs do not have to be of equal block size.

Suppose there are p designs for which the treatments can be divided into v_1 groups of v_2 distinct treatments each. We show that if $\mathbf{N}_h$ for $h = 1,..., p_1$ are incidence matrices of p_1 balanced block (BB) designs and $\mathbf{N}_h$ for

$h = p_1 + 1, \dots, p$ are incidence matrices of $p - p_1$ partially balanced block designs with group divisible associate scheme (GD PBIB) design with $\lambda_2 = \lambda_1 + 1$ (or with $\lambda_1 = \lambda_2 + 1 > 1$, $v_2 = 2$ and/or with $\lambda_2 = \lambda_1 + 2$, $v_1 = 2$), then a block design with the incidence matrix $\mathbf{N} = (\mathbf{N}_1, \mathbf{N}_2, \dots, \mathbf{N}_{p_1}, \mathbf{N}_{p_1+1}, \dots, \mathbf{N}_p)$ is an E-optimal partially balanced block design with group divisible associate scheme (GD PBB) design. The design with the incidence matrix **N** can have unequal replications and unequal block sizes.

2. Introduction

Let $\Omega_{v,k_1,\dots,k_b}$ denote the collection of all connected block designs with v treatments arranged in b blocks of size $k_1, \dots, k_b$, respectively. The well-known C-matrix (cf. e.g. Constantine, 1981), when design d is used, is defined by

$$\mathbf{C}_d = \text{diag}\{r_{d1}, \dots, r_{dv}\} - \mathbf{N}_d \text{ diag } \{k_{d1}^{-1}, \dots, k_{db}^{-1}\} \mathbf{N}'_d, \tag{2.1}$$

where r_{di} is the number of replications of the ith treatment in d ($i = 1, 2, \dots, v$), k_{dj} is the jth block size in d ($j = 1, 2, \dots, b$) and $\mathbf{N}_d = (n_{dij})$, with n_{dij} signifying how many times treatment i appears in block j. For a design $d \in \Omega_{v,k_1,\dots,k_b}$ let $0 = \mu_{d0} < \mu_{d1} \le \dots \le \mu_{dv-1}$ denote the eigenvalues of its C-matrix $\mathbf{C}_d$.

A design $d^* \in \Omega_{v,k_1,\dots,k_b}$ is called E-optimal if $\mu_{d^*1} = \mu_1^* \ge \mu_{d1}$ for all designs $d \in \Omega_{v,k_1,\dots,k_b}$. Note that μ_{d1} is the smallest positive eigenvalue of the $\mathbf{C}_d$ matrix. The following lemma (see Kiefer (1959), Ehrenfeld (1955), Constantine (1981)) gives statistical meaning to an E-optimal design.

Lemma 2.1. A design d^* is E-optimal if and only if the maximum variance among all best linear unbiased estimators of normalized linear contrasts of design d is smallest under d^*.

Definition 2.1. A GD PBIB design (see, e.g. Raghavarao, 1971) is a block design based on $v = v_1 v_2$ treatments (being arranged into v_1 groups of v_2 treatments each), consisting of b blocks of size k ($k<v$), such that each treatment occurs in r blocks. In the GD PBIB design, two treatments that belong to the same group are first associates and two treatments that belong to different groups are second associates. Two treatments which are i-th associates occur together in exactly λ_i blocks (i = 1, 2). Each treatment has exactly n_i i-th associates, where $n_1 = v_2 - 1$

and $n_2 = (v_1-1)v_2$. The numbers v, b, r, k, λ_1, λ_2, v_1, v_2 are called the parameters of the design.

If **N** is a $(v \times b)$ incidence matrix of a GD PBIB design, then from the above definition we have $\mathbf{NN'} = r\mathbf{A}_0 + \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2$, where $\mathbf{A}_0 = \mathbf{I}_v$, $\mathbf{A}_1 = \mathbf{I}_{v_1} \otimes (\mathbf{J}_{v_2} - \mathbf{I}_{v_2})$, $\mathbf{A}_2 = (\mathbf{J}_{v_1} - \mathbf{I}_{v_1}) \otimes \mathbf{J}_{v_2}$, while $\mathbf{I}_x$ is an identity matrix of order x, $\mathbf{J}_x$ is an $(x \times x)$-matrix of ones and $\otimes$ denotes the Kronecker product of matrices.

The construction of GD PBIB designs consists of finding a binary incidence matrix **N** satisfying the above condition.

The matrix $\mathbf{C}$ of a GD PBIB design can be expressed as:

$$\mathbf{C} = \mu_0\mathbf{P}_0 + \mu_1\mathbf{P}_1 + \mu_2\mathbf{P}_2, \tag{2.2}$$

where $\mu_0 = 0$, $\mu_1 = (r(k-1)+\lambda_1)k^{-1}$, $\mu_2 = v\lambda_2 k^{-1}$ are the eigenvalues of $\mathbf{C}$ with multiplicities $\alpha_0 = 1$, $\alpha_1 = v_1(v_2-1)$ and $\alpha_2 = v_1 - 1$, respectively, and

$$\mathbf{P}_0 = v^{-1}\mathbf{J}_v,\ \mathbf{P}_1 = \mathbf{I}_{v_1} \otimes (\mathbf{I}_{v_2} - v_2^{-1}\mathbf{J}_{v_2}),\ \mathbf{P}_2 = (\mathbf{I}_{v_1} - v_1^{-1}\mathbf{J}_{v_1}) \otimes v_2^{-1}\mathbf{J}_{v_2}.$$

A block design satisfying condition (2.2) is called a partially balanced block design with an association scheme of group divisible (GD PBB) design. It is known (cf. Kageyama 1974, Brzeskwiniewicz, 1989) that a GD PBB design with a constant block size has equal treatment replications and therefore it is a GD PBIB design. Thus, a GD PBIB design is a special case of GD PBB design.

Definition 2.2. A balanced incomplete block (BIB) design (see Raghavarao, 1971) is an arrangement of v treatments in b blocks of sizes k such that every treatment occurs r times and every pair of distinct treatments is contained in λ blocks. The numbers v, b, r, k and λ are called the parameters of the BIB design.

The C-matrix of a BIB design can be expressed as:

$$\mathbf{C} = \mu\left(\mathbf{I}_v - v^{-1}\mathbf{J}_v\right), \tag{2.3}$$

where $\mu = (n-b)(v-1)^{-1}$ with $n = vr = kb$.

A block design satisfying condition (2.3) is called balanced block design (BB) design. It is worth noting (cf. Kageyama 1974) that a BB design with a constant block size has equal treatment replications and therefore it is a BIB design. Thus a BIB design is a special case of BB design.

Kageyama (1974) has shown that for odd t matrices

$$\mathbf{N} = \begin{pmatrix} \mathbf{1}'_{(t+1)/2} & \mathbf{0}'_t & 1 \\ \mathbf{0}'_{(t+1)/2} & \mathbf{1}'_t & 1 \\ \mathbf{1}_t\mathbf{1}'_{(t+1)/2} & \mathbf{I}_t & \mathbf{0}_t \end{pmatrix}$$

and

$$\tilde{\mathbf{N}} = \begin{pmatrix} \mathbf{1}'_{t+2} & \mathbf{0}'_b \\ \mathbf{I}_{t+2} & \mathbf{N} \end{pmatrix} \tag{2.4}$$

are incidence matrices of BB designs with parameters:

$v = t+2$, $b = (t+1)/2+t+1$, $\mathbf{r} = \left((t+1)/2+1, t+1, ((t+1)/2+1)\mathbf{1}_t'\right)'$,
$\mathbf{k} = \left((t+1)\mathbf{1}'_{(t+1)/2}, 2\cdot\mathbf{1}'_{t+1}\right)'$ and $\tilde{v} = t+3$, $\tilde{b} = (t+1)/2+2t+3$,
$\tilde{\mathbf{r}} = \left(t+2, (t+1)/2+2, t+2, ((t+1)/2+2)\mathbf{1}_t'\right)'$,
$\tilde{\mathbf{k}} = \left(2\cdot\mathbf{1}'_{t+2}, (t+1)\mathbf{1}'_{(t+1)/2}, 2\cdot\mathbf{1}'_{t+1}\right)'$,

respectively, where $\mathbf{r}$ denotes the vector of treatment replications, $\mathbf{k}$ denotes the vector of block sizes, $\mathbf{1}_x$ ($\mathbf{1}'_x$) is a column (row) vector of x ones and $\mathbf{0}_x$ ($\mathbf{0}'_x$) is a column vector of x zeros.

3. Results

The following is a condition for E-optimality of GD PBB designs.

Theorem 3.1. Let $\mathbf{N}_h$ for $h = 1,2,\ldots,p_1$ be incidence matrices of BB designs with parameters v, b_h, $\mathbf{r}_h$, $\mathbf{k}_h$ and let $\mathbf{N}_h$ for $h = p_1+1, p_1+2,\ldots,p$ be incidence matrices of GD PBIB designs with parameters v, b_h, r_h, k_h, λ_{1h}, λ_{2h}. If the GD PBIB designs are E-optimal and $\overset{*}{\mu}_{h1}$ are the smallest positive eigenvalues of the matrices $\mathbf{C}_h$ and if there exists a vector $\mathbf{q} \neq \mathbf{0}$ such that $\mathbf{C}_h\mathbf{q} = \overset{*}{\mu}_{h1}\mathbf{q}$, then

$$\mathbf{N} = \left(\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_{p_1}, \mathbf{N}_{p_1+1}, \ldots, \mathbf{N}_p\right) \tag{3.1}$$

is the incidence matrix of an E-optimal GD PBB design d with parameters v,

$$b = \sum_{h=1}^{p} b_h, \; \mathbf{r} = \sum_{h=1}^{p_1} \mathbf{r}_h + \sum_{h=p_1+1}^{p} r_h \mathbf{1}_v,$$

$$\mathbf{k} = \left(\mathbf{k}'_1, \ldots, \mathbf{k}'_{p_1}, k_{p_1+1}\mathbf{1}'_{b_{p_1+1}}, \ldots, k_p \mathbf{1}'_{b_p}\right)', \text{ and } \mu_1^* = \sum_{h=1}^{p} \mu_{h1}^*.$$

Proof. From (3.1) and (2.1) we have $\mathbf{C} = \sum_{h=1}^{p} \mathbf{C}_h$, where $\mathbf{C}_h$ are defined in (2.3) for $h = 1,2,\ldots,p_1$ and in (2.2) for $h = p_1+1, p_1+2,\ldots,p$, respectively. Hence $\mathbf{C}_h\mathbf{q} = \mu_{h1}^*\mathbf{q}$ and $\mathbf{Cq} = \sum_{h=1}^{p} \mu_{h1}^*\mathbf{q} = \mu_1^*\mathbf{q}$. This completes the proof. ■

Jacroux (1980) showed that any GD PBIB design with

$$\lambda_2 = \lambda_1 + 1 \tag{3.2}$$

is an E-optimal design and

$$\mu_1^* = \mu_1 = \frac{r(k-1)+\lambda_1}{k}.$$

Jacroux (1983) showed also that any GD PBIB design with

$$\lambda_1 = \lambda_2 + 1 \text{ and } v_2 = 2 \tag{3.3}$$

is an E-optimal design with $\mu_1^* = \mu_1 = \dfrac{r(k-1)+\lambda_1}{k}$.

In Table 1, we present reference numbers of GD PBIB designs from the catalogue of Clatworthy (1973) satisfying (3.2) and/or (3.3).

Table 1. Reference number of GD PBIB designs with $\lambda_2=\lambda_1+1$ (i) or $\lambda_1=\lambda_2+1$ and $v_1=2$ (ii) from catalogue of the Clatworthy (1973)

reference	number	(i) or (ii)
R	3, 10, 18, 24, 27, 29, 34, 36, 38, 39, 40, 41, 46, 52, 54, 58, 62, 70, 78, 79, 81, 86, 88, 90, 91, 92, 93, 96, 101, 111, 112, 114, 117, 122, 125, 128, 129, 130, 134, 145, 150, 153, 161, 162, 163, 176, 183, 191, 201, 202, 205	(i)
	1, 6, 16, 20, 32, 43, 74, 94, 98, 141, 164	(ii)
SR	1, 6, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 23, 26, 28, 30, 31, 32, 33, 34, 35, 38, 41, 44, 46, 48, 49, 50, 51, 58, 60, 62, 63, 64, 65, 68, 71, 75, 77, 78, 79, 87, 88, 89, 90, 96, 97, 98, 104, 105, 110	(i)

From this and from Theorem 3.1 we obtain the following corollary.

Corollary 3.1. If in Theorem 3.1 the matrices $\mathbf{N}_h$, $h = p_1+1, p_2+2,\dots, p$ are incidence matrices of GD PBIB with (3.2) and/or with (3.3), then (3.1) is an incidence matrix of an E-optimal GD PBB design and

$$\mu_1^* = \sum_{h=1}^{p_1} \mu_{h1}^* + \sum_{h=p_1+1}^{p} \frac{r_h(k_h-1)+\lambda_{1h}}{k_h}.$$

Note that $\mathbf{q} \in C(\mathbf{P}_1)$, where $C(\mathbf{P}_1)$ is a column space of $\mathbf{P}_1$.

Cheng (1980) showed that any GD PBIB design with

$$\lambda_1 = \lambda_2 + 1 > 1 \text{ and } v_2 = 2 \tag{3.4}$$

is an E-optimal design with

$$\mu_1^* = \mu_2 = \frac{r(k-1)+\lambda_1-2}{k}.$$

In Table 2 we present reference numbers of GD PBIB designs from the catalogue of Clatworthy (1973) satisfying (3.4).

Table 2. Reference number of GD PBIB designs with $\lambda_1=\lambda_2+1$ and $v_2=2$ from catalogue of the Clatworthy (1973)

reference	number
R	1, 6, 16, 19, 30, 37, 42, 48, 71, 89, 97, 109, 132, 140, 186
S	1, 18, 51, 98

From this and from Theorem 3.1 the following results:

Corollary 3.2. If in Theorem 3.1 the matrices $\mathbf{N}_h$, $h = p_1+1, p_2+2, \ldots, p$ are incidence matrices of a GD PBIB design with (3.4), then (3.1) is an incidence matrix of an E-optimal GD PBB design with

$$\mu_1^* = \sum_{h=1}^{p_1} \mu_{h1}^* + \sum_{h=p_1+1}^{p} \frac{r_h(k_h-1)+\lambda_{1h}-2}{k_h}.$$ Note that $\mathbf{q} \in C(\mathbf{P}_2)$.

4. Example

From (2.4) with $t=3$, we take the incidence matrix

$$\mathbf{N}_1 = \begin{pmatrix} 1&1&1&1&1&0&0&0&0&0&0 \\ 1&0&0&0&0&1&1&0&0&0&1 \\ 0&1&0&0&0&0&0&1&1&1&1 \\ 0&0&1&0&0&1&1&1&0&0&0 \\ 0&0&0&1&0&1&1&0&1&0&0 \\ 0&0&0&0&1&1&1&0&0&1&0 \end{pmatrix}$$ of a BB design with parameters $v=6$,

$b_1=11$, $\mathbf{r}_1 = (5, 4, 5, 4\cdot\mathbf{1}_3')'$, $\mathbf{k}_1 = (2\cdot\mathbf{1}_5', 4\cdot\mathbf{1}_2', 2\cdot\mathbf{1}_4')'$ and $\mu_{11}^*=3$.

Matrices $$\mathbf{N}_2 = \begin{pmatrix} 1&1&0&0 \\ 0&0&1&1 \\ 1&0&1&0 \\ 0&1&0&1 \\ 1&0&0&1 \\ 0&1&1&0 \end{pmatrix}$$ and $$\mathbf{N}_3 = \begin{pmatrix} 1&0&0&1&0&0&1&0&0 \\ 1&0&0&0&1&0&0&0&1 \\ 0&1&0&0&1&0&0&1&0 \\ 0&1&0&0&0&1&1&0&0 \\ 0&0&1&0&0&1&0&0&1 \\ 0&0&1&1&0&0&0&1&0 \end{pmatrix}$$ are

incidence matrices of GD PBIB no. SR18 and SR6 from Table 1, respectively. For design SR18, we have $v=6$, $b_2=4$, $r_2=2$, $k_2=3$, $\lambda_{12}=0$,

$\lambda_{22}=1$, $\mu_{21}^{*}=1{,}33$ and for design SR6, we have $v=6$, $b_3=9$, $r_3=3$, $k_3=2$, $\lambda_{13}=0$, $\lambda_{23}=1$, $\mu_{31}^{*}=1{,}5$. From Corollary 3.1, it follows that $\mathbf{N}=(\mathbf{N}_1,\mathbf{N}_2,\mathbf{N}_3)$ is an incidence matrix of an E-optimal GD PBB design with parameters $v=6$, $b=24$, $\mathbf{r}=\mathbf{r}_1+5\cdot\mathbf{1}_6$, $\mathbf{k}=(\mathbf{k}_1',3\cdot\mathbf{1}_4',2\cdot\mathbf{1}_9')'$, $\mu_1^{*}=5{,}83$, i.e. $\mathbf{Cq}=\mu_1^{*}\mathbf{q}$ for any $\mathbf{q}\in C(\mathbf{P}_1)$.

5. Conclusion

In Theorem 3.1 we prove the E-optimality of several different types of block designs that have unequally replicated treatments and unequal block sizes. In statistical literature E-optimality of block design having only unequal block sizes (see i.e. Lee and Jacroux, 1987) and only unequally replicated treatments (see i.e. Jacroux, 1982). Theorem 3.1 can be applied to many designs which were previously known to be E-optimal.

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